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1998 J. Phys. A: Math. Gen. 31 2431

(http://iopscience.iop.org/0305-4470/31/10/017)

J. Phys. A: Math. Gen. **31** (1998) 2431–2446. Printed in the UK PII: S0305-4470(98)87975-4

Determinant structure of the rational solutions for the Painlevé IV equation

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Received 29 September 1997

Abstract. Rational solutions for the Painlevé IV equation are investigated by Hirota bilinear formalism. It is shown that the solutions in one hierarchy are expressed by 3-reduced Schur functions, and those in another two hierarchies by Casorati determinants of the Hermite polynomials, or by a special case of the Schur polynomials.

1. Introduction

It is known that the six Painlevé equations P_I–P_{VI} are the fundamental equations in the theory of nonlinear integrable systems in wider sense, and thus their solutions are regarded as the 'nonlinear version of special functions' [1]. Not only for the use of Painlevé equations in a physical context, they also have many interesting mathematical structures, one of which is the structure of particular solutions. As for the algebraic solutions, it is known that some of the Painlevé equations admit rational solutions expressed by classical polynomials, e.g. Jacobi and Legendre polynomials for P_{VI}, Laguerre polynomials for P_V and Hermite polynomials for P_{IV}. However, there also arise some non-classical polynomials, as pointed out by Umemura [2]. A typical example is the Yablonskii–Vorobe'v polynomials which appear in the rational solutions of P_{II},

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha. \tag{1}$$

Let T_m $(m = 0, 1, \ldots)$ be polynomials generated recursively by

$$\frac{d^2T_m}{dt^2}T_m - \left(\frac{dT_m}{dt}\right)^2 = \frac{1}{4}(tT_m^2 - T_{m+1}T_{m-1}) \qquad T_0 = T_1 = 1. \tag{2}$$

Then, it is known that

$$y = \frac{d}{dt}\log\frac{T_{m+1}}{T_m} \tag{3}$$

satisfies P_{II} with $\alpha = -m - 1$ [3, 4]. The characteristic polynomials T_m are called the Yablonskii–Vorobe'v polynomials. To clarify the nature of these polynomials, it is useful to study the relation with the soliton equations. In fact, P_{II} is derived from the similarity reduction of the modified KdV equation. This fact implies that the Yablonskii–Vorobe'v polynomials are closely related to the τ function which gives the rational solutions of the modified KdV equations. Based on this expectation, we have shown in the previous paper

that Yablonskii–Vorobe'v polynomials are expressed as the specialization of the 2-reduced Schur functions [5].

In this article, we investigate the rational solutions for $\mathrm{P_{IV}}$,

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w} \tag{4}$$

where α and β are parameters. There are various simple particular solutions of physical importance [6], and it is possible to obtain solutions of 'higher order' by applying the Bäcklund transformations [6–9] which map one solution to another solution of $\mathrm{P_{IV}}$ with different values of parameters. As for rational solutions of $\mathrm{P_{IV}}$(4), it is known that there are three hierarchies of unique rational solutions [7] (the name of the hierarchies are due to [10]),

'$-1/z$ hierarchy':

$$w = \frac{P_{n-1}(z)}{Q_n(z)} \qquad (\alpha,\beta) = (\pm k, -2(1+2l+k)^2) \qquad k,l \in \mathbb{Z},\ l \leqslant -1,\ k \leqslant -2l \tag{5}$$

'$-2z$ hierarchy':

$$w = -2z + \frac{P_{n-1}(z)}{Q_n(z)} \qquad (\alpha,\beta) = (k, -2(1+2l+k)^2) \qquad k,l \in \mathbb{Z},\ l \geqslant 0,\ k \leqslant -l \tag{6}$$

'$-\frac{2}{3}z$ hierarchy':

$$\begin{aligned} &w = -\frac{2}{3}z + \frac{P_{n-1}(z)}{Q_n(z)} \qquad (\alpha,\beta) = (2k, -2(\pm\tfrac{1}{3}+2l)^2) \\ &(2k+1, -2(\pm\tfrac{2}{3}+2l)^2) \qquad k,l \in \mathbb{Z} \end{aligned} \tag{7}$$

where $P_m(z)$ and $Q_m(z)$ are some polynomials in z of degree m, and that there are no other rational solutions.

Lukashevich has shown that the simplest solutions in the first two hierarchies are expressed by the Hermite polynomials [11]. Okamoto has studied the Bäcklund transformations and shown that the log derivative of ratio of two-directional Wronskians of the Hermite polynomials give the solutions of $\mathrm{P_{IV}}$ [12]. Moreover, Murata [7] has pointed out that any solution in the $-1/z$ hierarchy can be transformed to a solution in the $-2z$ hierarchy, and vice versa. This fact strongly implies that these two hierarchies have the same nature, and the solutions may be expressed by a log derivative of the ratio of some determinants whose entries are related to the Hermite polynomials. Those determinants are called the τ functions. We investigate these hierarchies by using the Hirota's bilinear formalism and show that all of the solutions in those hierarchies are expressed by the τ functions which is nothing but the Casorati determinants (or equivalently, the Wronskians) of the Hermite polynomials. Moreover, the τ functions are also expressible in terms of the Schur functions.

The structure of the solutions in $-\frac{2}{3}z$ hierarchy has been unknown, but Okamoto [12] has studied this hierarchy and obtained the following result.

Proposition 1.1 (Okamoto). Let Q_m, $m \in \mathbb{Z}_{\geqslant 0}$, be polynomials in x generated by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log Q_m + x^2 + 2m - 1 = \frac{Q_{m+1}Q_{m-1}}{Q_m^2} \qquad Q_0 = Q_1 = 1. \tag{8}$$

Then

$$u = \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{Q_{m+1}}{Q_m} - x \tag{9}$$

satisfies P_{IV},

$$\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 6xu + \frac{9}{2}\left(x^2 - \frac{4}{3}m\right)u - \frac{1}{2u}. \tag{10}$$

Similarly, let R_m, $m \in \mathbb{Z}_{\geqslant 0}$, be polynomials in x generated by

$$\frac{d^2}{dx^2}\log R_m + x^2 + 2m = \frac{R_{m+1}R_{m-1}}{R_m^2} \qquad R_0 = 1 \qquad R_1 = x. \tag{11}$$

Then

$$u = \frac{d}{dx}\log\frac{R_{m+1}}{R_m} - x \tag{12}$$

satisfies P_{IV},

$$\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 6xu + \frac{9}{2}\left(x^2 - \frac{4}{3}\left(m + \frac{1}{2}\right)\right)u - \frac{4}{2u}. \tag{13}$$

The characteristic polynomials Q_m and R_m are called the Okamoto polynomials [2]. Indeed, the solutions which are expressed by the Okamoto polynomials are a special case of the $-\frac{2}{3}z$ hierarchy.

The key to understanding the nature of the Okamoto polynomials lies in the relation with soliton equations, as in the case of P_{II}. In fact, several authors have pointed out the relation between P_{IV} and the Boussinesq equation [13–16] which belongs to the 3-reduction of the KP hierarchy [17]. This implies that some solutions of P_{IV} may be understood as the similarity reduction of the 3-reduced KP hierarchy. We show that the solutions in the $-\frac{2}{3}z$ hierarchy is indeed the case, namely they are expressed by the 3-reduced Schur functions.

2. $-\frac{2}{3}z$ Hierarchy

Let us consider the Schur functions in $x_1, x_2, \ldots$ labelled by the Young diagram $Y = (i_1, i_2, i_3, \ldots, i_l)$, $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_l \geqslant 0$,

$$S_Y(x_1, x_2, \ldots) = \begin{vmatrix} p_{i_1} & p_{i_1+1} & \cdots & p_{i_1+l-1} \\ p_{i_2-1} & p_{i_2} & \cdots & p_{i_2+l-2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{i_l-l+1} & p_{i_l-l+2} & \cdots & p_{i_l} \end{vmatrix} \tag{14}$$

where p_k's are polynomials in $x_1, x_2, \ldots$ defined by

$$\sum_{k=0}^{\infty} p_k(x_1, x_2, \ldots)\lambda^k = \exp\sum_{n=1}^{\infty} x_k\lambda^k \qquad p_n = 0 (n < 0). \tag{15}$$

It is well known that S_Y is a τ function of the KP hierarchy [17].

We consider the 3-reduction of the τ function, namely, we impose the condition,

$$\frac{\partial \tau_Y}{\partial x_{3k}} = 0 \qquad k = 1, 2, \ldots. \tag{16}$$

To realize this condition, it is sufficient to restrict the Young diagram as

$$Y = (M + 2n, M + 2n - 2, \ldots, M + 2, M^2, (M-1)^2, \ldots, 2^2, 1^2) \tag{17}$$

or

$$Y = (N + 2n - 1, N + 2n - 3, \ldots, N + 1, N^2, (N-1)^2, \ldots, 2^2, 1^2). \tag{18}$$

In fact, we can easily verify that the Schur functions associated with the above Young diagrams satisfy condition (16), by noting the relation,

$$\frac{\partial p_n}{\partial x_k} = p_{n-k}. \tag{19}$$

For notational simplicity, we rearrange the structure of the Schur polynomials associated with the Young diagrams (17) and (18) as

$$\tau_M^N(x_1, x_2, \ldots) = \left|\begin{array}{ccccc}
\cdots & p_{2N-1} & p_{2N} & p_{2N+1} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & p_{2N-3} & p_{2N-2} & p_{2N-1} & \cdots \\
 & & & \ddots & \\
\cdots\cdots\cdots\cdots\cdots & & p_3 & p_4 & p_5 \\
\cdots\cdots\cdots\cdots\cdots & & p_0 & p_1 & p_2 \\
\hline
\cdots & p_{2M-2} & p_{2M-1} & p_{2M} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & p_{2N-4} & p_{2N-3} & p_{2N-2} & \cdots \\
 & & & \ddots & \\
\cdots\cdots\cdots\cdots\cdots & & p_2 & p_3 & p_4 \\
\cdots\cdots\cdots\cdots\cdots & & p_{-1} & p_0 & p_1
\end{array}\right|. \tag{20}$$

We can see the equivalence of τ_M^N and 3-reduced Schur functions in the following manner,

$$\tau_M^N = \begin{cases} S_{(M+2n, M+2n-2, \ldots, M+2, M^2, (M-1)^2, \ldots, 1^2)} & \text{for } N = M + n > M \\ S_{(N+2n-1, N+2n-3, \ldots, N+1, N^2, (N-1)^2, \ldots, 1^2)} & \text{for } N < M = N + n. \end{cases} \tag{21}$$

Then, we have the following theorem.

Theorem 2.1. Let $x_1 = x$, $x_2 = \frac{1}{2}$, $x_k = 0 (k = 3, 4, \ldots)$ in τ_M^N. Then

$$w = \sqrt{\frac{2}{3}} \left\{ \left(\log \frac{\tau_{M+1}^N}{\tau_M^N} \right)_x - x \right\} \qquad z = \sqrt{\frac{3}{2}} x \tag{22}$$

satisfies $\mathrm{P_{IV}}(4)$ with

$$(\alpha, \beta) = \left(2M - N + 1, -2 \left(N + \frac{2}{3} \right)^2 \right) \qquad M, N \in \mathbb{Z}_{\geqslant 0}. \tag{23}$$

Similarly,

$$w = \sqrt{\frac{2}{3}} \left\{ \left(\log \frac{\tau_M^{N+1}}{\tau_M^N} \right)_x - x \right\} \qquad z = \sqrt{\frac{3}{2}} x \tag{24}$$

satisfies $\mathrm{P_{IV}}$ (4) with

$$(\alpha, \beta) = \left(-M + 2N + 2, -2 \left(M + \frac{1}{3} \right)^2 \right) \qquad M, N \in \mathbb{Z}_{\geqslant 0}. \tag{25}$$

Remark 2.2. The above solutions cover all of the $-\frac{2}{3}z$ hierarchy, which is easily verified by comparing the parameters (23) and (25) with (7).

Corollary 2.3. The Okamoto polynomials are the special case of 3-reduced Schur functions,

$$Q_m = \tau_0^m = S_{(2m, 2m-2, \ldots, 2)}(x, \tfrac{1}{2}, 0, \ldots) \qquad R_m = \tau_m^0 = S_{(2n-1, 2n-3, \ldots, 1)}(x, \tfrac{1}{2}, 0, \ldots). \tag{26}$$

Theorem 2.1 is proved by considering the similarity reduction of 3-reduction of the first modified KP hierarchy, namely, hierarchy of the Bäcklund transformations of the KP hierarchy. The first three of the bilinear equations in this hierarchy are given by [17],

$$(D_{x_1}^2 - D_{x_2})\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{27}$$

$$(D_{x_1}^3 + 3D_{x_1}D_{x_2})\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{28}$$

$$(D_{x_1}^4 - 3D_{x_1}^2 D_{x_2} - 6D_{x_2}^2)\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{29}$$

where D_x is the Hirota's bilinear differential operator defined by

$$D_x^n f \cdot g = (\partial_x - \partial_{x'})^n f(x)g(x')|_{x=x'}. \tag{30}$$

Next, we apply the similarity reduction.

Lemma 2.4. Let $\tau_N^M = \tau_N^M(x_1, x_2, 0, 0, \ldots)$. Then

$$\partial_{x_2}\tau_M^N = \frac{1}{2x_2}((M^2 - MN + N^2 + N)\tau_M^N - x_1\partial_{x_1}\tau_M^N). \tag{31}$$

Proof. Note that p_k, the entry of τ_M^N, is a homogeneous polynomial in $(x_1, x_2, x_3, \ldots)$ with degree k if we define the degree of x_k as k. Putting $x_k = 0$ $(k \geqslant 3)$, then τ_M^N is a homogeneous polynomial in x_1 and x_2 of degree $M^2 - MN + N^2 + N$. Thus, if we set

$$f_M^N = \frac{1}{x_2^{(M^2-MN+N^2+N)/2}}\tau_M^N \tag{32}$$

then f_M^N depends only on $t = \frac{x_1}{x_2^{1/2}}$. This implies

$$\partial_{x_2} f_M^N = \frac{\partial t}{\partial x_2}\frac{\mathrm{d}}{\mathrm{d}t} f_M^N = -\frac{1}{2}\frac{x_1}{x_2^{3/2}}\frac{\mathrm{d}}{\mathrm{d}t} f_M^N \tag{33}$$

$$\partial_{x_1} f_M^N = \frac{\partial t}{\partial x_1}\frac{\mathrm{d}}{\mathrm{d}t} f_M^N = \frac{1}{x_2^{1/2}}\frac{\mathrm{d}}{\mathrm{d}t} f_M^N \tag{34}$$

which yields

$$-2x_2\partial_{x_2} f_M^N = x_1\partial_{x_1} f_M^N. \tag{35}$$

Rewriting equation (35) in terms of τ_M^N, we obtain equation (31). □

By using lemma 2.4, all the x_2 derivatives in equations (27)–(29) are rewritten in terms of x_1 derivatives as

$$\left(D_{x_1}^2 + \frac{x_1}{2x_2}D_{x_1} - \frac{1}{2x_2}(2M - N + 1)\right)\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{36}$$

$$\left(D_{x_1}^3 - \frac{3}{2x_2}\left(-x_1 D_{x_1}^2 + (2M - N + 1)D_{x_1} + \partial_{x_1}\right)\right)\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{37}$$

$$\begin{aligned}\Bigg[D_{x_1}^4 - \frac{3}{2x_2}(-x_1 D_{x_1}^3 + (2M - N + 1)D_{x_1}^2 - 2\partial_{x_1}D_{x_1}) - \frac{6}{4x_2^2}(x_1^2 D_{x_1}^2 \\ -2x_1(2M - N + 1)D_{x_1} + 3x_1\partial_{x_1} - (3N^2 + 4N + 1))\Bigg]\tau_{M+1}^N \cdot \tau_M^N = 0\end{aligned} \tag{38}$$

respectively, where

$$\partial_{x_1} D_{x_1} f \cdot g = f_{x_1x_1}g - f g_{x_1x_1} \tag{39}$$

$$\partial_{x_1} f \cdot g = \partial_{x_1}(fg) = f_{x_1}g + f g_{x_1}. \tag{40}$$

Putting $x_1 = x$ and $x_2 = \frac{1}{2}$, we have the following proposition.

Proposition 2.5. $\tau_M^N = \tau_M^N(x, \frac{1}{2}, 0, \ldots)$ and $\tau_{M+1}^N = \tau_{M+1}^N(x, \frac{1}{2}, 0, \ldots)$ satisfy the following bilinear equations,

$$(D_x^2 + xD_x - (2M - N + 1))\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{41}$$

$$(D_x^3 - 3(-xD_x^2 + (2M - N + 1)D_x + \partial_x))\tau_{M+1}^N \cdot \tau_M^N = 0 \tag{42}$$

$$\begin{aligned}[D_x^4 + 3xD_x^3 - 3\{2x^2 + (2M - N + 1)\}D_x^2 + 12x(2M - N + 1)D_x \\ -6(-\partial_x D_x + 3x\partial_1 - (3N^2 + 4N + 1))]\tau_M^N \cdot \tau_{M+1}^N = 0.\end{aligned} \tag{43}$$

Now the proof of theorem 2.1 is straightforward. Dividing equations (41)–(43) by $\tau_M^N \tau_{M+1}^N$ and using the formulae [19],

$$\frac{D_x \tau_M^N \cdot \tau_{M+1}^N}{\tau_M^N \tau_{M+1}^N} = \phi_x \tag{44}$$

$$\frac{D_x^2 \tau_M^N \cdot \tau_{M+1}^N}{\tau_M^N \tau_{M+1}^N} = \rho_{xx} + \phi_x^2 \tag{45}$$

$$\frac{D_x^3 \tau_M^N \cdot \tau_{M+1}^N}{\tau_M^N \tau_{M+1}^N} = \phi_{xxx} + 3\phi_x \rho_{xx} + \phi_x^3 \tag{46}$$

$$\frac{D_x^4 \tau_M^N \cdot \tau_{M+1}^N}{\tau_M^N \tau_{M+1}^N} = \rho_{xxxx} + 4\phi_x \phi_{xxx} + 3\rho_{xx}^2 + 6\phi_x^2 \rho_{xx} + \phi_x^4 \tag{47}$$

$$\frac{\partial_x D_x \tau_M^N \cdot \tau_{M+1}^N}{\tau_M^N \tau_{M+1}^N} = \phi_{xx} + \phi_x \rho_x \tag{48}$$

where $\rho = \log(\tau_M^N \tau_{M+1}^N)$ and $\phi = \log(\tau_{M+1}^N / \tau_M^N)$, we obtain

$$\rho_{xx} + \phi_x^2 + x\phi_x - (2M - N + 1) = 0 \tag{49}$$

$$\phi_{xxx} + 3\phi_x \rho_{xx} + \phi_x^3 - 3x(\rho_{xx} + \phi_x^2) + 3(2M - N + 1)\phi_x - 3\rho_x = 0 \tag{50}$$

$$\begin{aligned}\rho_{xxxx} + 4\phi_x \phi_{xxx} + 3\rho_{xx}^2 + 6\phi_x^2 \rho_{xx} + \phi_x^4 + 3x(\phi_{xxx} + 3\phi_x \rho_{xx} + \phi_x^3) \\ -3(2M - N + 1 + 2x^2)(\rho_{xx} + \phi_x^2) + 12x(2M - N + 1)\phi_x \\ +6(\phi_{xx} + \phi_x \rho_x) - 18x\rho_x + 6(3N^2 + 4N + 1) = 0\end{aligned} \tag{51}$$

respectively. Eliminating ρ and putting $u = \phi_x - x$, we obtain $\mathrm{P_{IV}}$,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{1}{2u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{3}{2}u^3 + 6xu^2 + \left(\frac{9}{2}x^2 - 3a\right)u - \frac{3b + 1}{2u} \tag{52}$$

where,

$$a = 2M - N + 1 \qquad b = 3N^2 + 4N + 1. \tag{53}$$

This gives half of theorem 2.1. The other half is proved in a similar manner by starting from the bilinear equations,

$$(D_{x_1}^2 - D_{x_2})\tau_M^{N+1} \cdot \tau_M^N = 0 \tag{54}$$

$$(D_{x_1}^3 + 3D_{x_1} D_{x_2})\tau_M^{N+1} \cdot \tau_M^N = 0 \tag{55}$$

$$(D_{x_1}^4 - 3D_{x_1}^2 D_{x_2} - 6D_{x_2}^2)\tau_M^{N+1} \cdot \tau_M^N = 0 \tag{56}$$

from which we obtain equation (52) for $u = (\log \tau_M^{N+1}/\tau_M^N) - x$ with $a = -M + 2N + 2$ and $b = 3M^2 + 2M$. This completes the proof of theorem 2.1.

3. $-1/z$ and $-2z$ hierarchies

3.1. τ function

As mentioned in the introduction, solutions in the $-1/z$ and $-2z$ hierarchies can be transformed into each other by Bäcklund transformations. From the view of τ functions, this fact suggests that their τ functions are the same, and only the relation between the τ functions and the dependent variable of $\mathrm{P_{IV}}$ is different.

Definition 3.1. Let H_n and $\hat{H}_n$, $n = 0, 1, 2 \ldots$, be polynomials in x defined by

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n \lambda^n = \exp\left(x\lambda - \frac{1}{2}\lambda^2\right) \tag{57}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \hat{H}_n \lambda^n = \exp\left(x\lambda + \frac{1}{2}\lambda^2\right) \tag{58}$$

respectively. Then we define the τ functions τ_N^n and $\hat{\tau}_N^n$ to be $N \times N$ determinants given by

$$\tau_N^n = \begin{vmatrix} H_n & H_{n+1} & \cdots & H_{n+N-1} \\ H_{n+1} & H_{n+2} & \cdots & H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N-1} & H_{n+N} & \cdots & H_{n+2N-2} \end{vmatrix} \tag{59}$$

$$\hat{\tau}_N^n = \begin{vmatrix} \hat{H}_n & \hat{H}_{n+1} & \cdots & \hat{H}_{n+N-1} \\ \hat{H}_{n+1} & \hat{H}_{n+2} & \cdots & \hat{H}_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{H}_{n+N-1} & \hat{H}_{n+N} & \cdots & \hat{H}_{n+2N-2} \end{vmatrix} \tag{60}$$

respectively.

Then, the solutions in the $-1/z$ hierarchy are expressed as follows.

Theorem 3.2.

$$w = -\sqrt{2}\left(\log \frac{\tau_{N+1}^n}{\tau_N^n}\right)_x \qquad z = \frac{1}{\sqrt{2}}x \tag{61}$$

give rational solutions of $\mathrm{P_{IV}}(4)$ with

$$(\alpha, \beta) = (-(n + 2N + 1), -2n^2) \qquad n, N \in \mathbb{Z} \qquad n \geqslant 1 \qquad N \geqslant 0. \tag{62}$$

Moreover,

$$w = \sqrt{2}\left(\log \frac{\hat{\tau}_{N+1}^n}{\hat{\tau}_N^n}\right)_x \qquad z = \frac{1}{\sqrt{2}}x \tag{63}$$

give rational solutions of $\mathrm{P_{IV}}(4)$ with

$$(\alpha, \beta) = (n + 2N + 1, -2n^2) \qquad n, N \in \mathbb{Z} \qquad n \geqslant 1 \qquad N \geqslant 0. \tag{64}$$

Here, several remarks are in order.

Remark 3.3.

(1) Parametrization equation (62) is equivalent to equation (5) with $\alpha = -k$ if we put $k = n + 2N + 1$ and $l = -(N + 1)$. Moreover, equation (64) is equivalent to equation (5) with $\alpha = k$ if we put $k = n+2N+1$ and $l = -(N+1)$. Hence the solutions in theorem 3.2 cover all of $-1/z$ hierarchy.

(2) As shown in the proof, n do not necessarily have to be integers and hence H_n can be the Hermite–Weber functions. In fact, we need only the recursion relations,

$$\frac{\mathrm{d}}{\mathrm{d}x}H_n - xH_n = -H_{n+1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}H_n = nH_{n-1}. \tag{65}$$

Moreover, τ function admits several expressions.

Remark 3.4.

(1) It is possible to express the solutions in terms of the Wronskian, since we have the relation,

$$\begin{vmatrix} H_n & H_{n+1} & \cdots & H_{n+N-1} \\ H_{n+1} & H_{n+2} & \cdots & H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N-1} & H_{n+N} & \cdots & H_{n+2N-2} \end{vmatrix} = \prod_{k=1}^{N-1}(n+k)^k$$
$$\times \begin{vmatrix} H_{n+N-1} & \frac{\mathrm{d}}{\mathrm{d}x}H_{n+N-1} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N-1}H_{n+N-1} \\ \frac{\mathrm{d}}{\mathrm{d}x}H_{n+N-1} & \frac{\mathrm{d}^2}{\mathrm{d}x^2}H_{n+N-1} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N}H_{n+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N-1}H_{n+N-1} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{N}H_{n+N-1} & \cdots & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2N-2}H_{n+N-1} \end{vmatrix} \tag{66}$$

which can be verified by using equation (65).

(2) As is obvious from the Wronskian expression equation (66), τ_N^0 only gives a constant, which yields a 0 solution of $\mathrm{P_{IV}}$. Thus only the cases of $n \geqslant 1$ give nontrivial solutions.

(3) τ functions are also expressed by the Schur functions. In fact we have, for example,

$$\hat{\tau}_N^n = \frac{(n+N-1)^!}{(n-1)^!}(-1)^{N(N-1)/2}S_{(n^N)}(x, \tfrac{1}{2}, 0, \ldots) \tag{67}$$

where,

$$k^! = k!^k(k-1)!^{(k-1)}\ldots 2!^2 1!^1$$

which can be verified by noting

$$\hat{H}_n = \frac{1}{n!}p_n(x, \tfrac{1}{2}, 0, \ldots) \tag{68}$$

and equation (66).

For the solutions in the $-2z$ hierarchy, the following expression is valid.

Theorem 3.5.

$$w = \sqrt{2}\left\{\left(\log\frac{\tau_{N+1}^n}{\tau_N^{n+1}}\right)_x - x\right\} \qquad z = \frac{1}{\sqrt{2}}x \tag{69}$$

give rational solutions of $\mathrm{P_{IV}}$ (4) with

$$(\alpha, \beta) = (N-n, -2(n+N+1)^2) \qquad n, N \in \mathbb{Z} \qquad n \geqslant 0 \qquad N \geqslant 0. \tag{70}$$

Remark 3.6.

(1) Parametrization equation (70) is equivalent to equation (6), if we put $k = N - n$, $l = N$. Hence the solutions in theorem 3.5 covers all the solutions in the $-2z$ hierarchy.

(2) Another τ function $\hat{\tau}_N^n$ (60) can give the solution of the same type for $\mathrm{P_{IV}}$. In fact, we can show that

$$\hat{w} = -\sqrt{2}\left\{\log\left(\frac{\hat{\tau}_{N+1}^n}{\hat{\tau}_N^{n+1}}\right)_x - x\right\} \qquad z = \frac{1}{\sqrt{2}}x \tag{71}$$

satisfies P_{IV} with the parameters $(\alpha, \beta) = (n-N, -2(n+N+1)^2)$. This parametrization is also equivalent to equation (6), if we put $k = n-N$, $l = n$. However, the uniqueness of the rational solutions [7, 18] implies that they give the same solution as given in theorem 3.5. In fact, we can check that w and $\hat{w}$ are the same if we exchange n and N.

3.2. Proof of theorems

In this section, we give the proof of the theorems 3.2 and 3.5. The first half of theorem 3.2 is a direct consequence of the following proposition.

Proposition 3.7. τ_N^n satisfies the following bilinear equations.

$$(D_x^2 - xD_x + n)\tau_{N+1}^n \cdot \tau_N^n = 0 \tag{72}$$

$$\left(D_x^3 - 6xD_x^2 + (5x^2 + n - 4N - 2)D_x + \frac{\mathrm{d}}{\mathrm{d}x} - 5nx\right)\tau_{N+1}^n \cdot \tau_N^n = 0 \tag{73}$$

$$\begin{aligned}\Big(&D_x^4 - 4xD_x^3 + (11x^2 - 8n - 28N - 14)D_x^2 \\ &-2\frac{\mathrm{d}}{\mathrm{d}x}D_x - 2x(4x^2 - 3n - 20N - 10)D_x \\ &- 3x\frac{\mathrm{d}}{\mathrm{d}x} + 8nx^2 - n(9n + 28N + 14)\Big)\tau_{N+1}^n \cdot \tau_N^n = 0.\end{aligned} \tag{74}$$

The second half of theorem 3.2 is derived from the following bilinear equations.

Proposition 3.8. $\hat{\tau}_N^n$ satisfies the following bilinear equations.

$$(D_x^2 + xD_x - n)\hat{\tau}_{N+1}^n \cdot \hat{\tau}_N^n = 0 \tag{75}$$

$$\left(D_x^3 + 6xD_x^2 + (5x^2 - n + 4N + 2)D_x - \frac{\mathrm{d}}{\mathrm{d}x} - 5nx\right)\hat{\tau}_{N+1}^n \cdot \hat{\tau}_N^n = 0 \tag{76}$$

$$\begin{aligned}\Big(&D_x^4 + 4xD_x^3 + (11x^2 + 8n + 28N + 14)D_x^2 \\ &+2\frac{\mathrm{d}}{\mathrm{d}x}D_x + 2x(4x^2 + 3n + 20N + 10)D_x \\ &-3x\frac{\mathrm{d}}{\mathrm{d}x} - 8nx^2 - n(9n + 28N + 14)\Big)\hat{\tau}_{N+1}^n \cdot \hat{\tau}_N^n = 0.\end{aligned} \tag{77}$$

Finally, theorem 3.5 is obtained from the following bilinear equations.

Proposition 3.9. τ_N^n satisfies the following bilinear equations.

$$(D_x^2 - xD_x + n - N)\tau_{N+1}^n \cdot \tau_N^{n+1} = 0 \tag{78}$$

$$\left(D_x^3 - 6xD_x^2 + (5x^2 + n - N)D_x + \frac{\mathrm{d}}{\mathrm{d}x} - 5(n-N)x\right)\tau_{N+1}^n \cdot \tau_N^{n+1} = 0 \tag{79}$$

$$\begin{aligned}\Big(&D_x^4 - 4xD_x^3 + (11x^2 - 8n - 28N - 18)D_x^2 + 2\frac{\mathrm{d}}{\mathrm{d}x}D_x \\ &-2x(4x^2 - 3n - 15N - 9)D_x - x\frac{\mathrm{d}}{\mathrm{d}x} + 8(n-N)x^2 \\ &-(9n^2 + 10Nn + 14n - 27N^2 - 22N)\Big)\tau_{N+1}^n \cdot \tau_N^{n+1} = 0.\end{aligned} \tag{80}$$

To prove propositions 3.7–3.9, we first introduce the notation τ_{NY}^n.

Definition 3.10. Let $Y = (i_1, i_2, \ldots, i_h)$ be a Young diagram. Then we define an $N \times N$ determinant τ^n_{NY} by

$$\tau^n_{NY} = \begin{vmatrix} H_n & H_{n+1} & \cdots & H_{n+N-h-1} & H_{n+N-h+i_h} & \cdots & H_{n+N-2+i_2} & H_{n+N-1+i_1} \\ H_{n+1} & H_{n+2} & \cdots & H_{n+N-h} & H_{n+N-h+1+i_h} & \cdots & H_{n+N-1+i_2} & H_{n+N+i_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ H_{n+N-1} & H_N & \cdots & H_{n+2N-h-2} & H_{n+2N-h-1+i_h} & \cdots & H_{n+2N-3+i_2} & H_{n+2N-2+i_1} \end{vmatrix}. \tag{81}$$

It is possible to derive the bilinear equations from the Plücker relations which are identities between the determinants whose columns are shifted. In fact, proposition 3.7 is obtained from the following identities.

Lemma 3.11.

$$\tau^n_{N+1\ydiagram{1,1}}\tau^n_N - \tau^n_{N+1\ydiagram{1}}\tau^n_{N\ydiagram{1}} + \tau^n_{N+1}\tau^n_{N\ydiagram{2}} = 0 \tag{82}$$

$$\tau^n_{N+1\ydiagram{2,1}}\tau^n_N - \tau^n_{N+1\ydiagram{2}}\tau^n_{N\ydiagram{1}} + \tau^n_{N+1}\tau^n_{N\ydiagram{3}} = 0 \tag{83}$$

$$\tau^n_{N+1}\tau^n_{N\ydiagram{4}} - \tau^n_{N+1\ydiagram{3}}\tau^n_{N\ydiagram{1}} + \tau^n_N\tau^n_{N+1\ydiagram{2,1}} = 0 \tag{84}$$

$$\tau^n_{N+1\ydiagram{1}}\tau^n_{N\ydiagram{3}} - \tau^n_{N+1\ydiagram{2}}\tau^n_{N\ydiagram{2}} + \tau^n_N\tau^n_{N+1\ydiagram{2,2}} = 0. \tag{85}$$

Proposition 3.8 is derived from the same identity in $\hat{\tau}^n_N$. Similarly, we obtain proposition 3.9 from the following identities.

Lemma 3.12.

$$\tau^n_{N+1\ydiagram{1,1}}\tau^{n+1}_N - \tau^n_{N+1\ydiagram{1}}\tau^{n+1}_{N\ydiagram{1}} + \tau^n_{N+1}\tau^{n+1}_{N\ydiagram{2}} = 0 \tag{86}$$

$$\tau^n_{N+1\ydiagram{2,1}}\tau^{n+1}_N - \tau^n_{N+1\ydiagram{2}}\tau^{n+1}_{N\ydiagram{1}} + \tau^n_{N+1}\tau^{n+1}_{N\ydiagram{3}} = 0 \tag{87}$$

$$\tau^n_{N+1}\tau^{n+1}_{N\ydiagram{4}} - \tau^n_{N+1\ydiagram{3}}\tau^{n+1}_{N\ydiagram{1}} + \tau^n_N\tau^{n+1}_{N+1\ydiagram{2,1}} = 0 \tag{88}$$

$$\tau^n_{N+1\ydiagram{1}}\tau^{n+1}_{N\ydiagram{3}} - \tau^n_{N+1\ydiagram{2}}\tau^{n+1}_{N\ydiagram{2}} + \tau^{n+1}_N\tau^n_{N+1\ydiagram{2,2}} = 0. \tag{89}$$

We give the derivations of lemmas 3.11 and 3.12 in the appendix.

We next construct the shift operators which are differential operators generating τ^n_{NY} from τ^n_N by using the technique developed in [20, 5]. In fact, we have the following lemma.

Lemma 3.13.

$$\tau^n_{N\ydiagram{1}} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + xN\right)\tau^n_N. \tag{90}$$

Proof. Note that $\tau^n_{N\ydiagram{1}}$ is expressed by

$$\tau^n_{N\ydiagram{1}} = \begin{pmatrix} H_{n+1} & H_{n+2} & \cdots & H_{n+N} \\ H_{n+2} & H_{n+3} & \cdots & H_{n+N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N} & H_{n+N+1} & \cdots & H_{n+2N-1} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix} \tag{91}$$

where Δ_{ij} is the (i, j)-cofactor of τ^n_N and $A \cdot B$ denotes a standard scalar product for $N \times N$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ which is defined as

$$A \cdot B = \sum_{i,j=1}^{N} a_{ij} b_{ij} = \mathrm{Tr}\, A^t B. \tag{92}$$

The first matrix of (91) is rewritten by using the recursion relation (65) as

$$-\begin{pmatrix} \partial_x H_n & \partial_x H_{n+1} & \cdots & \partial_x H_{n+N-1} \\ \partial_x H_{n+1} & \partial_x H_{n+2} & \cdots & \partial_x H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x H_{n+N-1} & \partial_x H_{n+N} & \cdots & \partial_x H_{n+2N-2} \end{pmatrix} + x \begin{pmatrix} H_n & H_{n+1} & \cdots & H_{n+N-1} \\ H_{n+1} & H_{n+2} & \cdots & H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N-1} & H_{n+N} & \cdots & H_{n+2N-2} \end{pmatrix}. \tag{93}$$

Then applying the dot product to equation (93), we obtain

$$\tau^n_{N\square} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + xN\right)\tau^n_N. \tag{94}$$

Thus we have proved lemma 3.13. □

For the shift operators of second order, we have the following.

Lemma 3.14.

$$\tau^n_{N\square\square} = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (2N+1)x\frac{\mathrm{d}}{\mathrm{d}x} + x^2 N(N-1) - N(N+n+1)\right)\tau^n_N \tag{95}$$

$$\tau^n_{N\substack{\square\\\square}} = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (2N-1)x\frac{\mathrm{d}}{\mathrm{d}x} + N(N-1)x^2 + N(N+n-1)\right)\tau^n_N. \tag{96}$$

Proof. We consider

$$\tau^n_{N\square\square} + \tau^n_{N\substack{\square\\\square}} = \begin{pmatrix} H_{n+1} & H_{n+2} & \cdots & H_{n+N-1} & H_{n+N+1} \\ H_{n+2} & H_{n+3} & \cdots & H_{n+N} & H_{n+N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{n+N} & H_{n+N+1} & \cdots & H_{n+2N-2} & H_{n+2N} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{\square 11} & \Delta_{\square 12} & \cdots & \Delta_{\square 1N} \\ \Delta_{\square 21} & \Delta_{\square 22} & \cdots & \Delta_{\square 2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{\square N1} & \Delta_{\square N2} & \cdots & \Delta_{\square NN} \end{pmatrix} \tag{97}$$

where $\Delta_{\square ij}$ is (i, j) cofactor of $\tau_{N\square}$. The first matrix in the right-hand side is equal to

$$-\begin{pmatrix} \partial_x H_n & \partial_x H_{n+1} & \cdots & \partial_x H_{n+N-2} & \partial_x H_{n+N} \\ \partial_x H_{n+1} & \partial_x H_{n+2} & \cdots & \partial_x H_{n+N-1} & \partial_x H_{n+N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_x H_{n+N-1} & \partial_x H_{n+N} & \cdots & \partial_x H_{n+2N-1} & \partial_x H_{n+2N-1} \end{pmatrix} + x \begin{pmatrix} H_n & H_{n+1} & \cdots & H_{n+N-2} & H_{n+N} \\ H_{n+1} & H_{n+2} & \cdots & H_{n+N-1} & H_{n+N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{n+N-1} & H_{n+N} & \cdots & H_{n+2N-1} & H_{n+2N-1} \end{pmatrix}. \tag{98}$$

Applying the dot product to equation (98), we obtain,

$$\tau^n_{N\square\square} + \tau^n_{N\square\!\!\!\!\square} = \left(-\frac{\mathrm{d}}{\mathrm{d}x} + xN\right)\tau^n_{N\square}$$
$$= \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2xN\frac{\mathrm{d}}{\mathrm{d}x} - N + x^2N^2\right)\tau^n_N. \tag{99}$$

Next we consider the following equality,

$$\tau^n_{N\square\square} - \tau^n_{N\square\!\!\!\!\square} = \begin{pmatrix} H_{n+2} & H_{n+3} & \cdots & H_{n+N+1} \\ H_{n+3} & H_{n+4} & \cdots & H_{n+N+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N+1} & H_{n+N+2} & \cdots & H_{n+2N} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix}. \tag{100}$$

The first matrix of the right hand side of (100) is rewritten as

$$-\begin{pmatrix} \partial_x H_{n+1} & \partial_x H_{n+2} & \cdots & \partial_x H_{n+N} \\ \partial_x H_{n+2} & \partial_x H_{n+3} & \cdots & \partial_x H_{n+N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x H_{n+N} & \partial_x H_{n+N+1} & \cdots & \partial_x H_{n+2N-1} \end{pmatrix}$$
$$+x\begin{pmatrix} H_{n+1} & H_{n+2} & \cdots & H_{n+N} \\ H_{n+2} & H_{n+3} & \cdots & H_{n+N+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n+N} & H_{n+N+1} & \cdots & H_{n+2N-1} \end{pmatrix}. \tag{101}$$

The first matrix of equation (101) is rewritten as

$$\begin{pmatrix} (n+1)H_n & (n+2)H_{n+1} & \cdots & (n+N)H_{n+N-1} \\ (n+2)H_{n+1} & (n+3)H_{n+2} & \cdots & (n+N+1)H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ (n+N)H_{n+N-1} & (n+N+1)H_{n+N} & \cdots & (n+2N-1)H_{n+2N-2} \end{pmatrix}$$
$$= \begin{pmatrix} (n+1)H_n & (n+1)H_{n+1} & \cdots & (n+1)H_{n+N-1} \\ (n+2)H_{n+1} & (n+2)H_{n+2} & \cdots & (n+2)H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ (n+N)H_{n+N-1} & (n+N)H_{n+N} & \cdots & (n+N)H_{n+2N-2} \end{pmatrix}$$
$$+\begin{pmatrix} 0 & H_{n+1} & \cdots & (N-1)H_{n+N-1} \\ 0 & H_{n+2} & \cdots & (N-1)H_{n+N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & H_{n+N} & \cdots & (N-1)H_{n+2N-2} \end{pmatrix}. \tag{102}$$

Applying the dot product to equation (102), we obtain

$$[\{(n+1)+\cdots+(n+N)\}+1+\cdots+(N-1)]\tau^n_N = N(N+n)\tau^n_N.$$

Moreover, from the second term of equation (101), we have

$$x\tau_{N\square} = x\left(-\frac{\mathrm{d}}{\mathrm{d}x} + xN\right)\tau^n_N.$$

Finally, we obtain

$$\tau^n_{N\square\square} - \tau^n_{N\square\!\!\!\!\square} = \left(-x\frac{\mathrm{d}}{\mathrm{d}x} + x^2N - N(N+n)\right)\tau^n_N. \tag{103}$$

From equations (99) and (103), we obtain equations (95) and (96). Thus we have proved lemma 3.14. □

Continuing the similar but tedious calculations, we obtain the following shift operators of third order.

Lemma 3.15.

$$\tau^n_{N\square\square\square} = \frac{1}{6}\Big[-\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3(N+1)x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (-(3N^2+6N+2)x^2 + 3N^2 + 3nN + 7N + 2n + 3)\frac{\mathrm{d}}{\mathrm{d}x} + Nx((N+1)(N+2)x^2 - 3nN - 3N^2 - 4n - 9N - 6)\Big]\tau^n_N \tag{104}$$

$$\tau^n_{N\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} = \frac{1}{3}\Big[-\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3Nx\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ((-3N^2+1)x^2 - n + N)\frac{\mathrm{d}}{\mathrm{d}x} + Nx((N^2-1)x^2 + 2n)\Big]\tau^n_N \tag{105}$$

$$\tau^n_{N\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} = \frac{1}{6}\Big[-\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3x(N-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ((-3N^2+6N-2)x^2 - 3nN + 2n - 3N^2 + 7N - 3)\frac{\mathrm{d}}{\mathrm{d}x} + Nx((N^2-3N+2)x^2 + 3nN - 4n + 3N^2 - 9N + 6)\Big]\tau^n_N. \tag{106}$$

Now we are ready to prove the proposition 3.7. Substituting the shift operators in lemmas 3.13 and 3.14 into the identities (82) and (83) in lemma 3.11, we obtain equations (72) and (73). Equation (74) is obtained in similar manner by using the shift operators of fourth order, which will be given in the appendix. This completes the proof of proposition 3.7. Proposition 3.9 is proved from the identities in lemma 3.12. Finally, to prove proposition 3.8, it is necessary to calculate the shift operators for $\hat{\tau}^n_N$. We omit the details, but this is done by simply replacing the recursion relations equation (65) by

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{H}_n + x\hat{H}_n = \hat{H}_{n+1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\hat{H}_n = n\hat{H}_{n-1}. \tag{107}$$

Thus the proof of theorems 3.2 and 3.5 is completed.

4. Concluding remarks

In this article, we have investigated the hierarchies of rational solutions for $\mathrm{P_{IV}}$, and shown that:

(1) solutions in $-\frac{2}{3}z$ hierarchy are expressed in terms of 3-reduced Schur functions. In particular, the Okamoto polynomials are nothing but their special cases;

(2) solutions in $-2z$ and $-1/z$ hierarchies are expressed in terms of a Casorati determinant of the Hermite polynomials. Moreover, they are also expressed by special cases of the Schur functions.

It might be an important and interesting problem to characterize the non-classical polynomials which appears in the algebraic solutions of the Painlevé equations listed in [2] by studying the determinant expressions with the aid of results of the soliton theory.

We finally note that after obtaining the results, the authors were informed that Noumi and Yamada have independently obtained the same expressions for the rational solutions of

P_{IV} [21]. They adopted some symmetric expression for P_{IV} and vertex operator approach for proof, while we have used only 'brute force', namely, a determinantal technique. Indeed the former approach is more elegant than the latter, but we did not use it since we plan to work on discrete cases also. Actually, we do not have elegant tools to deal with solutions of discrete Painlevé equations yet.

Acknowledgments

The authors are grateful to Professor B Grammaticos, Professor A Ramani and Professor J Hietarinta for discussions and encouragement. They also thank Professor H Umemura and Professor K Okamoto for their interest in this work and discussions. KK was supported by the Grant-in-aid for Encouragement of Young Scientist, The Ministry of Education, Science, Sports and Culture of Japan, no 09740164.

Appendix A. Derivation of Plücker relations

We consider the following identity of $(2N+2)\times(2N+2)$ determinant,

$$0 = \left|\begin{array}{cccc:ccc:ccc} 0 & 1 & \dots & N-1 & & \emptyset & & N & N+1 & \phi_1 \\ \hdashline & & \emptyset & & 0 & \dots & N-2 & N & N+1 & \phi_1 \end{array}\right| \tag{108}$$

where 'k' denotes the column vector,

$$\text{'}k\text{'} = \begin{pmatrix} H_{n+k} \\ H_{n+k+1} \\ \vdots \\ H_{n+k+N-1} \end{pmatrix} \tag{109}$$

and

$$\phi_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} . \tag{110}$$

Applying the Laplace expansion on the right-hand side of equation (108), we obtain

$$\begin{aligned} 0 = {} & |0,\dots,N-2,N-1,N| \times |0,\dots N-2,N+1,\phi_1| - |0,\dots,N-2,N-1,N+1| \\ & \times |0,\dots N-2,N,\phi_1| + |0,\dots,N-2,N-1,\phi_1| \\ & \times |0,\dots N-2,N,N+1| = \tau^n_{N+1}\tau^n_{N\square\square} - \tau^n_{N+1\square}\tau^n_{N\square} + \tau^n_N\tau^n_{N+1\square\square} \end{aligned} \tag{111}$$

which is nothing but equation (82). Similarly, equation (83) is derived from the following identity,

$$0 = \left|\begin{array}{cccc:ccc:ccc} 0 & 1 & \dots & N-1 & & \emptyset & & N & N+2 & \phi_1 \\ \hdashline & & \emptyset & & 0 & \dots & N-2 & N & N+2 & \phi_1 \end{array}\right| . \tag{112}$$

Moreover, we have the following higher-order identities:

$$\begin{aligned} 0 &= \left|\begin{array}{cccc:ccc:ccc} 0 & 1 & \dots & N-1 & & \emptyset & & N & N+3 & \phi_1 \\ \hdashline & & \emptyset & & 0 & \dots & N-2 & N & N+3 & \phi_1 \end{array}\right| \\ &= |0,\dots,N-2,N-1,N| \times |0,\dots N-2,N+3,\phi_1| \end{aligned}$$

$$
\begin{aligned}
&-|0,\ldots,N-2,N-1,N+3|\times|0,\ldots N-2,N,\phi_1| \\
&+|0,\ldots,N-2,N-1,\phi_1|\times|0,\ldots N-2,N,N+3| \\
&=\tau^n_{N+1}\tau^n_{N\square\square\square}-\tau^n_{N+1\square\square\square}\tau^n_{N\square}+\tau^n_N\tau^n_{N+1\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
0 &= \left|\begin{array}{c:c:c}
0 \quad 1 \quad \ldots \quad N-1 & \emptyset & N+1 \quad N+2 \quad \phi_1 \\ \hdashline
\emptyset & 0 \quad \ldots \quad N-2 & N+1 \quad N+2 \quad \phi_1
\end{array}\right| \\
&=|0,\ldots,N-2,N-1,N+1|\times|0,\ldots N-2,N+2,\phi_1| \\
&\quad -|0,\ldots,N-2,N-1,N+2|\times|0,\ldots N-2,N+1,\phi_1| \\
&\quad +|0,\ldots,N-2,N-1,\phi_1|\times|0,\ldots N-2,N+1,N+2| \\
&=\tau^n_{N+1\square}\tau^n_{N\square\square\square}-\tau^n_{N+1\square\square}\tau^n_{N\square\square}+\tau^n_N\tau^n_{N+1\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}.
\end{aligned}
\tag{114}
$$

Identities between τ^n_{N+1Y} and τ^{n+1}_{NY} are derived only by replacing ϕ_1 in the above identities by

$$
\phi_2=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}. \tag{115}
$$

Appendix B. List of the shift operators

$$
\begin{aligned}
\tau^n_{N\square\square\square\square}=\frac{1}{24}\Bigg[&\frac{\mathrm{d}^4}{\mathrm{d}x^4}-2(2N+3)x\frac{\mathrm{d}^3}{\mathrm{d}x^3}+\{(6N^2+18N+11)x^2-6nN-8n-6N^2 \\
&-22N-18\}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\{(-4N^3-18N^2-22N-6)x^2+12N^3+12nN^2 \\
&+58N^2+30nN+12n+78N+25\}\frac{\mathrm{d}}{\mathrm{d}x}+N\{(N^3+6N^2+11N+6)x^4 \\
&-(6nN^2+22nN+18n+6N^3+36N^2+66N+36)x^2+3N^3+3n^2N \\
&+6nN^2+18N^2+6n^2+24nN+22n+33N+18\}\Bigg]\tau^n_N
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\tau^n_{N\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}=\frac{1}{8}\Bigg[&\frac{\mathrm{d}^4}{\mathrm{d}x^4}-2x(2N+1)\frac{\mathrm{d}^3}{\mathrm{d}x^3}+\Big\{(6N^2+6N-1)x^2-2nN-2N^2-6N-2\Big\}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\
&+x\{(-4N^3-6N^2+2N+2)x^2+4nN^2-2nN-4n+4N^3+10N^2 \\
&+2N-3\}\frac{\mathrm{d}}{\mathrm{d}x}+N\{(N^3+2N^2-N-2)x^4+(-2N^3-4N^2-2nN^2 \\
&+2nN+6n+2N+4)x^2-N^3+n^2N-2nN^2-2N^2-2n^2-4nN \\
&-2n+N+2\}\Bigg]\tau^n_N
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
\tau^n_{N\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}=\frac{1}{12}\Bigg[&\frac{\mathrm{d}^4}{\mathrm{d}x^4}-4Nx\frac{\mathrm{d}^3}{\mathrm{d}x^3}+\{(6N^2-1)x^2+4n+2N\}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\{2N(-2N^2+1)x^2 \\
&-6nN-2N^2+1\}\frac{\mathrm{d}}{\mathrm{d}x}+N\{N(N^2-1)x^4+2nNx^2+3N^3+3n^2N \\
&+6nN^2-2n-3N\}\Bigg]\tau^n_N
\end{aligned}
\tag{118}
$$

$$\tau^n_{N\begin{smallmatrix}\square\square\\ \square\\ \square\end{smallmatrix}} = \frac{1}{8}\Bigg[\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2x(-2N+1)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \{(6N^2-6N-1)x^2 + 2nN + 2N^2 - 6N + 2\}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$
$$+x\{(-4N^3+6N^2+2N-2)x^2 - 4N^3 - 4nN^2 + 10N^2 - 2nN + 4n$$
$$-2N-3\}\frac{\mathrm{d}}{\mathrm{d}x} + \{(N^3-2N^2-N+2)x^4 + (2nN^2+2nN-6n+2N^3$$
$$-4N^2-2N+4)x^2 - n^2N + 2n^2 - 2nN^2 + 4nN - 2n - N^3$$
$$+2N^2+N-2\}\Bigg]\tau^n_N \tag{119}$$

$$\tau^n_{N\begin{smallmatrix}\square\square\\ \square\\ \square\\ \square\end{smallmatrix}} = \frac{1}{24}\Bigg[\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2x(-2N+3)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \{(6N^2-18N+11)x^2 + 6N^2 + 6nN - 22N$$
$$-8n+18\}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\{(-4N^3+18N^2-22N+6)x^2 - 12N^3 - 12nN^2$$
$$+58N^2 + 30nN - 12n - 78N + 25\}\frac{\mathrm{d}}{\mathrm{d}x} + N\{(N^3-6N^2+11N-6)x^4$$
$$+(6N^3+6nN^2-36N^2-22nN+18n+66N-36)x^2 + 3N^3 + 3n^2N$$
$$+6nN^2 - 18N^2 - 6n^2 - 24nN + 22n + 33N - 18\}\Bigg]\tau^n_N. \tag{120}$$

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